

Leibniz Central Extension on the Twisted Schrödinger-Virasoro Algebra¹

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Abstract. In this paper we present all the Leibniz 2-cocycles of the twisted Schrödinger-Virasoro algebra \mathcal{L} , which determine the second Leibniz cohomology group of \mathcal{L} .

Key words: Schrödinger-Virasoro algebras; Leibniz 2-cocycles; Leibniz cohomology group.

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1. Introduction

Motivated by the research for the free Schrödinger equations, the original Schrödinger-Virasoro Lie algebra was introduced in [3], in the context of non-equilibrium statistical physics. The infinite-dimensional Lie algebra considered in this paper is called the twisted Schrödinger-Virasoro algebra, which is the twisted deformation of the original Schrödinger-Virasoro Lie algebra. Both original and twisted sectors are closely related to the Schrödinger Lie algebras and the Virasoro Lie algebra, which both play important roles in many areas of mathematics and physics (e.g., statistical physics) and have been investigated in a series of papers (see [4], [5] and [14]–[18]).

Now we give the definition of the Lie algebra \mathcal{L} . A Lie algebra \mathcal{L} is called a *twisted Schrödinger-Virasoro Lie algebra* (see [13]), if \mathcal{L} has the \mathbb{C} -basis

$$\{L_n, Y_n, M_n \mid n \in \mathbb{Z}\}$$

with the Lie brackets (others vanishing)

$$[L_n, L_{n'}] = (n' - n)L_{n+n'}, \quad [L_n, M_p] = pM_{n+p}, \quad (1.1)$$

$$[L_n, Y_m] = (m - \frac{n}{2})Y_{n+m}, \quad [Y_m, Y_{m'}] = (m' - m)M_{m+m'}. \quad (1.2)$$

The twisted Schrödinger-Virasoro Lie algebra has an infinite-dimensional *twisted Schrödinger subalgebra* denoted by SS with the \mathbb{C} -basis $\{Y_n, M_n \mid n \in \mathbb{Z}\}$ and a *Virasoro subalgebra* denoted by $\mathcal{V}ir$ with the \mathbb{C} -basis $\{L_n, C \mid n \in \mathbb{Z}\}$.

The Schrödinger-Virasoro Lie algebras have recently drawn some attentions in the literature. Particularly, the sets of generators provided by the cohomology classes of the cocycles for both original and twisted sectors were presented in [13], and the derivation algebra and the automorphism group of the twisted sector were determined in [11]. Furthermore, vertex algebra representations of these Lie algebras were constructed in [18], and the irreducible

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modules with finite-dimensional weight spaces and indecomposable modules over them were considered in [12].

The main purpose of this paper is to determine the Leibniz 2-cocycles and further the second Leibniz cohomology group of twisted Schrödinger-Virasoro Lie algebra \mathcal{L} defined above. It is well known that the 2-cocycles on Lie algebras play important roles in the central extensions of Lie algebras, which can be used to construct many infinite dimensional Lie algebras and in particular, all 1-dimensional central extensions of \mathcal{L} are determined by the 2-cohomology group of \mathcal{L} . So do the Leibniz 2-cocycles and Leibniz cohomology groups of Lie algebras or further Leibniz algebras. Therefore, there appeared a number of papers on Leibniz 2-cocycles and Leibniz cohomology groups of infinite dimensional Lie algebras and Leibniz algebras (see [6], [8]–[10], [19] and related references cited in them). Now let's formulate our main results below.

We start with a brief definition. A *Leibniz algebra* L over \mathbb{C} is a vector space equipped with a \mathbb{C} -bilinear map $[\cdot, \cdot] : L \times L \longrightarrow L$ satisfying the Leibniz identity:

$$[x, [y, z]] = [[x, y], z] - [[x, z], y], \quad \forall x, y, z \in L. \quad (1.3)$$

It is easy to see that a Lie algebra must be a Leibniz algebra, while a Leibniz algebra over \mathbb{C} gives rise to be a Lie algebra if $[x, x] = 0$ for any $x \in L$.

Recall that a *Leibniz 2-cocycle* on L is a \mathbb{C} -bilinear function $\psi : L \times L \longrightarrow \mathbb{C}$ satisfying the Jacobian identity:

$$\psi(x, [y, z]) = \psi([x, y], z) - \psi([x, z], y) \quad (1.4)$$

for $x, y, z \in L$. In order to distinguish the Leibniz 2-cocycles from the usual 2-cocycles (which are anti-symmetric in addition), we call the usual ones Lie 2-cocycles. Denote by $C_L^2(L, \mathbb{C})$ the vector space of Leibniz 2-cocycles on L . For any \mathbb{C} -linear function $f : L \longrightarrow \mathbb{C}$, one can define a Leibniz 2-cocycle ψ_f as follows

$$\psi_f(x, y) = f([x, y]), \quad \forall x, y \in L, \quad (1.5)$$

which is called a *Leibniz 2-coboundary* or a *trivial Leibniz 2-cocycle* on L . Denote by $B_L^2(L, \mathbb{C})$ the vector space of Leibniz 2-coboundaries on L . A Leibniz 2-cocycle ϕ is said to be *equivalent* to another Leibniz 2-cocycle ψ if $\phi - \psi$ is trivial. For a 2-cocycle ψ , we denote by $[\psi]$ the equivalent class of ψ . The quotient space

$$H_L^2(L, \mathbb{C}) = C_L^2(L, \mathbb{C}) / B_L^2(L, \mathbb{C}) = \{\text{the equivalent classes of 2-cocycles}\}, \quad (1.6)$$

is called the *second Leibniz cohomology group* of L .

Theorem 1.1. *The second Leibniz cohomology group of L , $H_L^2(\mathcal{L}, \mathbb{C}) \cong \mathbb{C}$ is generated by the Virasoro Leibniz cocycle (see (2.6)).*

Throughout the article, we denote by \mathbb{Z}^* the set of all nonzero integers and \mathbb{C}^* the set of all nonzero complex numbers.

2. Proof of the main results

Let ψ be any Leibniz 2-cocycle. Our main attempt or method is to subtract all equivalent classes of the Leibniz 2-coboundaries on \mathcal{L} from ψ . The proof of Theorem 1.1 will be based on several technical lemmas and divided into three cases with some subcases.

According to the brackets (1.1)–(1.2), it is easy to see that the following identities hold:

$$L_n = \begin{cases} \frac{1}{n}[L_0, L_n] & \text{if } n \neq 0, \\ \frac{1}{2}[L_{-1}, L_1] & \text{if } n = 0, \end{cases} \quad M_n = \begin{cases} \frac{1}{n}[L_0, M_n] & \text{if } n \neq 0, \\ [L_{-1}, M_1] & \text{if } n = 0, \end{cases} \quad Y_n = \begin{cases} \frac{1}{n}[L_0, Y_n] & \text{if } n \neq 0, \\ \frac{2}{3}[L_{-1}, Y_1] & \text{if } n = 0. \end{cases}$$

Define a \mathbb{C} -linear function $f : \mathcal{L} \rightarrow \mathbb{C}$ as follows

$$\begin{aligned} f(Y_0) &= \frac{2}{3}\psi(L_{-1}, Y_1), \quad f(L_0) = \frac{1}{2}\psi(L_{-1}, L_1), \quad f(M_0) = \psi(L_{-1}, M_1), \quad \text{while} \\ f(Y_n) &= \frac{1}{n}\psi(L_0, Y_n), \quad f(L_n) = \frac{1}{n}\psi(L_0, L_n), \quad f(M_n) = \frac{1}{n}\psi(L_0, M_n), \quad \text{if } n \in \mathbb{Z}^*. \end{aligned}$$

Let $\varphi = \psi - \psi_f$ where ψ_f is defined in (1.5). One has

$$\varphi(L_{-1}, L_1) = \varphi(L_{-1}, Y_1) = \varphi(L_{-1}, M_1) = 0, \quad (2.1)$$

$$\varphi(L_0, L_n) = \varphi(L_0, Y_n) = \varphi(L_0, M_n) = 0, \quad \forall n \in \mathbb{Z}^*. \quad (2.2)$$

Lemma 2.1. *The following identities hold:*

$$\varphi(L_1, L_{-1}) = \varphi(L_1, Y_{-1}) = \varphi(L_1, M_{-1}) = 0, \quad (2.3)$$

$$\varphi(L_n, L_0) = \varphi(Y_n, L_0) = \varphi(M_n, L_0) = 0, \quad \forall n \in \mathbb{Z}^*. \quad (2.4)$$

Proof. Replacing ψ by φ and y by x in (1.4) simultaneously, one has

$$\varphi(x, [x, z]) + \varphi([x, z], x) = 0, \quad \forall x, z \in \mathcal{L}. \quad (2.5)$$

For any $n \in \mathbb{Z}^*$, replacing (x, z) by (L_n, L_{-n}) , (L_n, Y_{-n}) , (L_n, M_{-n}) in (2.5) respectively, we obtain the following identity:

$$\varphi(L_0, L_n) + \varphi(L_n, L_0) = \varphi(L_n, Y_0) + \varphi(Y_0, L_n) = \varphi(L_n, M_0) + \varphi(M_0, L_n) = 0,$$

which together with (2.2) gives (2.4).

Using the Jacobian identity on the three triples (L_1, L_1, L_{-2}) , (L_1, L_1, Y_{-2}) , (L_1, L_1, M_{-2}) respectively, one has

$$\varphi(L_{-1}, L_1) + \varphi(L_1, L_{-1}) = \varphi(L_{-1}, Y_1) + \varphi(L_1, Y_{-1}) = \varphi(L_{-1}, M_1) + \varphi(L_1, M_{-1}) = 0,$$

which together with (2.1) gives (2.3). Then the lemma follows. \square

Lemma 2.2. *For any $m, n \in \mathbb{Z}$, one can write*

$$\varphi(L_n, L_m) = \frac{n^3 - n}{12} \delta_{m, -n}. \quad (2.6)$$

Proof. For any $m, n \in \mathbb{Z}$, applying the Jacobian identity on the two triples (L_0, L_{-n}, L_n) , (L_0, L_m, L_n) and (L_m, L_n, L_{-m-n}) , one immediately has

$$\begin{aligned} n(2\varphi(L_0, L_0) + \varphi(L_n, L_{-n}) + \varphi(L_{-n}, L_n) &= 0, \\ (m+n)\varphi(L_m, L_n) + (n-m)\varphi(L_{m+n}, L_0) &= 0, \end{aligned}$$

which together with (2.4), give

$$(m+n)\varphi(L_m, L_n) = \varphi(L_0, L_0) = \varphi(L_n, L_{-n}) + \varphi(L_{-n}, L_n) = 0. \quad (2.7)$$

For any $m, n \in \mathbb{Z}$, applying the Jacobian identity on the triple (L_m, L_n, L_{-m-n}) , we obtain

$$(m-1)\varphi(L_{m+1}, L_{-m-1}) = (m+2)\varphi(L_m, L_{-m}), \quad (2.8)$$

in which using induction on n , one can write

$$\varphi(L_n, L_{-n}) = \frac{n^3 - n}{12}, \quad \forall n \in \mathbb{C}. \quad (2.9)$$

Then the lemma follows. \square

Lemma 2.3. *For any $m, n \in \mathbb{Z}$, one has*

$$\varphi(M_m, M_n) = \varphi(Y_m, M_n) = 0.$$

Proof. For any $m, n \in \mathbb{Z}$, applying the Jacobian identity on the three triples (L_1, M_{-1}, M_0) , (Y_0, Y_m, M_n) and (Y_{-n}, Y_m, M_n) respectively, one has

$$\varphi(M_0, M_0) = m\varphi(M_m, M_n) = (m+n)\varphi(M_{m-n}, M_n) = 0,$$

which immediately gives

$$\varphi(M_m, M_n) = 0 \quad \forall m, n \in \mathbb{Z}.$$

For any $m, n \in \mathbb{Z}$, applying the Jacobian identity on the three triples (L_1, Y_{-1}, M_0) , (L_0, Y_n, M_{-n}) and (L_m, Y_0, M_n) , one has

$$n(\varphi(M_{-n}, Y_n) + \varphi(Y_n, M_{-n})) = 0, \quad (2.10)$$

$$2n\varphi(M_{m+n}, Y_0) + m\varphi(Y_m, M_n) = 0, \quad (2.11)$$

which immediately gives

$$n\varphi(M_n, Y_0) = \varphi(Y_n, M_0) = 0 \quad \forall n \in \mathbb{Z}. \quad (2.12)$$

and further implies

$$m(m+n)\varphi(Y_m, M_n) = 0 \quad \forall n \in \mathbb{Z}. \quad (2.13)$$

For any $n \in \mathbb{Z}$, applying the Jacobian identity on the two triples (L_n, Y_0, M_{-n}) and (Y_n, Y_0, Y_{-n}) , one has

$$\begin{aligned} n(\varphi(Y_n, M_{-n}) - 2\varphi(M_0, Y_0)) &= 0, \\ n(\varphi(M_n, Y_{-n}) - \varphi(Y_n, M_{-n}) - 2\varphi(M_0, Y_0)) &= 0, \end{aligned}$$

which gives

$$\varphi(M_n, Y_{-n}) = 2\varphi(Y_n, M_{-n}) = 4\varphi(M_0, Y_0) \quad \forall n \in \mathbb{Z}^*,$$

and together with (2.10), further infers

$$\varphi(M_n, Y_{-n}) = \varphi(Y_n, M_{-n}) = 0 \quad \forall n \in \mathbb{Z}.$$

Then the lemma follows. □

Lemma 2.4. *For any $m, n \in \mathbb{Z}$, one has*

$$\varphi(L_n, M_m) = \varphi(M_n, L_m) = nc_1\delta_{m,-n}, \quad \varphi(Y_n, Y_m) = 2nc_1\delta_{m,-n}$$

for some constant $c_1 \in \mathbb{C}$.

Proof. For any $m, n \in \mathbb{Z}$, applying the Jacobian identity on the three triples (L_m, L_0, M_n) , (L_1, L_{-1}, M_0) and (L_m, L_0, M_n) respectively, one has

$$\begin{aligned} n\varphi(L_0, M_{m+n}) + n\varphi(M_n, L_m) - m\varphi(L_m, M_n) &= 0, \\ \varphi(L_0, M_0) = (m+n)\varphi(L_m, M_n) + n\varphi(M_{m+n}, L_0) &= 0, \end{aligned}$$

which together with (2.2) and (2.4), immediately give ($\forall m, n \in \mathbb{Z}$)

$$n\varphi(M_n, L_m) = m\varphi(L_m, M_n), \quad (2.14)$$

$$(m+n)\varphi(L_m, M_n) = \varphi(M_0, L_0) = \varphi(L_0, M_0) = 0. \quad (2.15)$$

For any $n \in \mathbb{Z}$, applying the Jacobian identity on the triple (L_n, L_1, M_{-n-1}) , one has

$$(n-1)\varphi(L_{n+1}, M_{-n-1}) = (n+1)(\varphi(L_n, M_{-n}) - \varphi(L_1, M_{-1})),$$

which gives

$$\varphi(L_n, M_{-n}) = \begin{cases} \frac{1}{2}n(n-1)\varphi(L_2, M_{-2}) + n(2-n)\varphi(L_1, M_{-1}) & \text{if } n \geq 0, \\ \frac{1}{6}n(n-1)\varphi(L_{-2}, M_2) + \frac{1}{3}n(n+2)\varphi(L_1, M_{-1}) & \text{if } n \leq -1. \end{cases} \quad (2.16)$$

Applying the Jacobian identity on the triple (L_1, L_{-2}, M_1) , one has

$$3\varphi(L_{-1}, M_1) = \varphi(L_{-2}, M_2) - \varphi(L_1, M_{-1}),$$

which together with the case $n = -1$ of (2.16) gives

$$\varphi(L_{-2}, M_2) = 3\varphi(L_2, M_{-2}) - 8\varphi(L_1, M_{-1}),$$

and together with which, (2.16) becomes

$$\varphi(L_n, M_{-n}) = n(n-1)c_1 + n(n-2)c_2 \quad \forall n \in \mathbb{Z}, \quad (2.17)$$

by denoting $\varphi(L_1, M_{-1}) = -c_2$, $\varphi(L_2, M_{-2}) = 2c_1$. Using (2.14), (2.15) and (2.17), we obtain

$$\varphi(M_n, L_{-n}) = -n(n+1)c_1 - n(n+2)c_2 \quad \forall n \in \mathbb{Z}, \quad (2.18)$$

For any $m, n \in \mathbb{Z}$, applying the Jacobian identity on the two triples (L_0, Y_m, Y_n) and (L_m, Y_0, Y_n) , one has

$$\begin{aligned} (n-m)\varphi(L_0, M_{m+n}) + n\varphi(Y_n, Y_m) - m\varphi(Y_m, Y_n) &= 0, \\ 2n\varphi(L_m, M_n) + m\varphi(Y_m, Y_n) + (2n-m)\varphi(Y_{m+n}, Y_0) &= 0, \end{aligned} \quad (2.19)$$

which together with (2.2) and (2.15), immediately give

$$n\varphi(Y_n, Y_m) = m\varphi(Y_m, Y_n) \quad \forall m, n \in \mathbb{Z}, \quad (2.20)$$

which further infers

$$n\varphi(Y_n, Y_0) = n(\varphi(Y_n, Y_{-n}) + \varphi(Y_{-n}, Y_n)) = 0 \quad \forall n \in \mathbb{Z}. \quad (2.21)$$

Then recalling (2.17), (2.18), (2.20) and (2.21), one can rewrite (2.19) as

$$2n\varphi(L_m, M_n) + n\varphi(Y_n, Y_m) + 3n\varphi(Y_{m+n}, Y_0) = 0,$$

from which one can obtain the following identity:

$$\varphi(Y_n, Y_m) = -2n((n+1)c_1 + (n+2)c_2)\delta_{m,-n} - 3\varphi(Y_0, Y_0)\delta_{m,-n} \quad \forall m, n \in \mathbb{Z}^*. \quad (2.22)$$

Then noticing the following identities:

$$\begin{aligned} \varphi(Y_n, Y_{-n}) &= -2n((n+1)c_1 + (n+2)c_2) - 3\varphi(Y_0, Y_0) \quad n \in \mathbb{Z}^*, \\ \varphi(Y_{-n}, Y_n) &= 2n((-n+1)c_1 + (-n+2)c_2) - 3\varphi(Y_0, Y_0) \quad n \in \mathbb{Z}^*, \end{aligned}$$

and recalling (2.21), one has

$$\begin{aligned} & 2n((-n+1)c_1 + (-n+2)c_2) - 3\varphi(Y_0, Y_0) \\ &= 2n((n+1)c_1 + (n+2)c_2) + 3\varphi(Y_0, Y_0), \end{aligned}$$

which gives

$$3\varphi(Y_0, Y_0) + 2(c_1 + c_2)n^2 = 0 \quad \forall n \in \mathbb{Z}^*,$$

and further forces

$$\varphi(Y_0, Y_0) = c_1 + c_2 = 0. \quad (2.23)$$

Then using (2.23), one can rewrite (2.17), (2.18) and (2.22) respectively as follows:

$$\varphi(Y_n, Y_{-n}) = 2\varphi(L_n, M_{-n}) = 2\varphi(M_n, L_{-n}) = 2nc_1 \quad \forall n \in \mathbb{Z}.$$

Then the lemma follows. \square

Lemma 2.5. *For any $m, n \in \mathbb{Z}$, one has*

$$\varphi(L_m, Y_n) = \varphi(Y_n, L_m) = 0.$$

Proof. For any $m, n \in \mathbb{Z}$, applying the Jacobian identity on the three triples (L_m, L_0, Y_0) , (L_m, L_0, Y_n) , (L_m, L_n, Y_0) and (L_0, L_m, Y_n) , one has

$$\begin{aligned} & m(2\varphi(L_m, Y_0) - \varphi(Y_m, L_0)) = 0, \\ & 2(m+n)\varphi(L_m, Y_n) - (m-2n)\varphi(Y_{m+n}, L_0) = 0, \\ & n\varphi(L_m, Y_n) + 2(n-m)\varphi(L_{m+n}, Y_0) + m\varphi(Y_m, L_n) = 0, \\ & 2m\varphi(L_m, Y_n) - 2n\varphi(Y_n, L_m) + (m-2n)\varphi(L_0, Y_{m+n}) = 0, \end{aligned}$$

which further can be rewritten as follows (recalling (2.2) and (2.4)):

$$\varphi(L_m, Y_0) = 0 \quad \forall m \in \mathbb{Z}^*, \quad \varphi(L_m, Y_n) = 0 \quad \forall m+n \in \mathbb{Z}^*, \quad (2.24)$$

$$\varphi(L_{-n}, Y_n) = \varphi(Y_{-n}, L_n) - 3\varphi(L_0, Y_0) \quad \forall n \in \mathbb{Z}^*, \quad (2.25)$$

$$\varphi(L_{-n}, Y_n) = -\varphi(Y_n, L_{-n}) - \frac{3}{2}\varphi(L_0, Y_0) \quad \forall n \in \mathbb{Z}^*. \quad (2.26)$$

The identities (2.25), (2.26) together with (2.1) and (2.3) force

$$\varphi(Y_{-1}, L_1) = 3\varphi(L_0, Y_0) = 0, \quad (2.27)$$

which together with (2.25) and (2.26) gives

$$\varphi(L_n, Y_{-n}) + \varphi(L_{-n}, Y_n) = 0. \quad (2.28)$$

For any $n \in \mathbb{Z}$, applying the Jacobian identity on the triple (L_n, L_1, Y_{-n-1}) and recalling (2.27), one has

$$2(n-1)\varphi(L_{n+1}, Y_{-1-n}) = (2n+3)\varphi(L_n, Y_{-n}),$$

which gives (by using induction on n)

$$\varphi(L_n, Y_{-n}) = 0 \quad \forall n \in \mathbb{Z}. \quad (2.29)$$

which together with (2.25) further forces

$$\varphi(Y_n, L_{-n}) = 0 \quad \forall n \in \mathbb{Z}. \quad (2.30)$$

Then this lemma follows from (2.24), (2.29) and (2.30). \square

Lemma 2.6. *One can suppose*

$$c_1 = 0.$$

Proof. Define another \mathbb{C} -linear function $g : \mathcal{L} \rightarrow \mathbb{C}$ as follows

$$g(M_0) = c_1, \quad \text{other components vanishing.}$$

Still denote $\varphi - \psi_g$ by φ where ψ_g is defined in (1.5). One has

$$c_1 = \varphi(L_{-1}, M_1) = 0. \quad (2.31)$$

Proof of Theorem 1.1 The theorem follows by the series of lemmas from the second one to the last one.

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